Proof of Theorem 1 (iii). We will prove the result for any $d \geq 3$.

**Theorem.** Let $y = (y_1, \ldots, y_d)$ be a binary vector with mean $p = (p_1, \ldots, p_d) \in (0,1)^d$, and a structured correlation matrix $R$. If the structure of $R$ is AR(1) with parameter $\rho$, then a joint distribution for $y$ exists if and only if

$$\max_{1 \leq i \leq (d-1)} L(p_i, p_{i+1}) \leq \rho \leq \min_{1 \leq i \leq (d-1)} U(p_i, p_{i+1}), \quad \text{(A2)}$$

where $L(a, b)$ and $U(a, b)$ are as defined on page 198 in Chaganty and Joe (2006).

**Proof.** The necessity is obvious. Sufficiency follows from the results in Qaqish (Biometrika, 2003, pp. 455-463). It is easy to check that if (A2) holds then $\lambda_i$ given by (6) in Qaqish (2003, p. 458) will lie in the interval $[0, 1]$ for $i = 2, \ldots, d$. Hence there exists a joint distribution for $y = (y_1, \ldots, y_d)$ in the conditional linear family.

A direct argument for the sufficiency is based on Markov chains of order 1 for binary time series discussed in Joe (1997, pp. 246-248). Let $q_i = 1 - p_i$ and $\sigma_i = (p_i q_i)^{1/2}$. Assume that (A2) holds. Then

$$H_i = \begin{bmatrix}
0 & y_{i+1} \\
q_{i+1} + \rho \frac{\sigma_{i+1}}{q_i} & p_{i+1} - \rho \frac{\sigma_{i+1}}{q_i} \\
q_{i+1} - \rho \frac{\sigma_{i+1}}{p_i} & p_{i+1} + \rho \frac{\sigma_{i+1}}{p_i}
\end{bmatrix}$$

is a transition matrix for $i = 1, 2, \ldots, (d - 1)$. We can construct a joint distribution for $y = (y_1, \ldots, y_d)$ explicitly using a first order Markov chain. Let $y_1$ be Bernoulli with mean $p_1$. For $i \geq 1$, assume that the transition from $y_i$ to $y_{i+1}$ is governed by the transition matrix $H_i$. It is easy to check that the marginal of $y_i$ is Bernoulli with mean $p_i$, and $\text{Corr}(y_i, y_j) = \rho^{|i-j|}$ for all $1 \leq i, j \leq d$.

Note that, in the above, the joint distributions for $y = (y_1, \ldots, y_d)$ obtained by the conditional linear family and by the first order Markov chain, are identical.