Appendix 1

Example where the working correlation is infeasible for binary responses
The covariates are same as those in Liang and Zeger (1986).

		Repeated measurements				
Subject		time1	time2	time3	time4	time5
1	binary response(y)	1	1	1	1	0
	covariate (x)	0.2	0.4	0.6	0.8	1
2	binary response(y)	1	0	0	0	0
	covariate (x)	0.2	0.4	0.6	0.8	1
3	binary response(y)	1	1	1	1	0
	covariate (x)	0.2	0.4	0.6	0.8	1
4	binary response(y)	1	0	0	0	0
	covariate (x)	0.2	0.4	0.6	0.8	1
5	binary response(y)	1	1	1	1	0
	covariate (x)	0.2	0.4	0.6	0.8	1
6	binary response(y)	1	1	1	1	0
	covariate (x)	0.2	0.4	0.6	0.8	1
7	binary response(y)	0	0	0	0	0
	covariate (x)	0.2	0.4	0.6	0.8	1
8	binary response(y)	1	1	1	1	0
	covariate (x)	0.2	0.4	0.6	0.8	1
9	binary response(y)	1	1	1	1	0
	covariate (x)	0.2	0.4	0.6	0.8	1
10	binary response(y)	0	0	0	0	0
	covariate (x)	0.2	0.4	0.6	0.8	1
11	binary response(y)	1	1	1	1	0
	covariate (x)	0.2	0.4	0.6	0.8	1
12	binary response(y)	1	1	1	1	0
	covariate (x)	0.2	0.4	0.6	0.8	1
13	binary response(y)	1	1	1	1	0
	covariate (x)	0.2	0.4	0.6	0.8	1
14	binary response(y)	1	1	1	1	0
	covariate (x)	0.2	0.4	0.6	0.8	1
15	binary response(y)	1	1	1	1	0
	covariate (x)	0.2	0.4	0.6	0.8	1

Logit Model: logit (E(y)) =b0+b1*x

GEE Estimates: PROC GENMOD in SAS: b0= 3.9036, b1= --4.6380.
Compound symmetry working correlation estimate of rho=0.4906,

wheras feasible interval for rho is [-0.0811, 0.1564].

Probit Model: E(y)=Φ(b0+b1*x), where Φ is the standard normal cdf

GEE Estimates: PROC GENMOD in SAS: b0=2.0673, b1= --2.6165.
Compound symmetry working correlation estimate of rho=0.4673,

whereas feasible interval for rho is [-0.1089, 0.1639].

In both cases the working correlation is outside the feasible region, and thus some bivariate probabilities will be negative.

Appendix 2

LEMMA **1.** Let $\Sigma_{n\times n}$ be positive definite and $D_{n\times p}$ be of rank p. Then

$$\Sigma - D(D'\Sigma^{-1}D)^{-1}D'$$

is nonnegative definite.

Proof: Let Y be a random vector such that $Cov(Y) = \Sigma$ and let $\hat{Y} = D(D'\Sigma^{-1}D)^{-1}D'\Sigma^{-1}Y$. We can check that $Cov(\hat{Y}, Y) = Cov(\hat{Y}) = D(D'\Sigma^{-1}D)^{-1}D'$. Hence,

$$Cov(Y - \hat{Y}) = Cov(Y) - Cov(\hat{Y}) = \Sigma - D(D'\Sigma^{-1}D)^{-1}D'$$

is nonnegative definite. $\square$

LEMMA **2.** For any matrix $B_{n\times p}$ of rank p we have

$$(B'D)^{-1}\ B'\Sigma B\ (D'B)^{-1} - (D'\Sigma^{-1}D)^{-1}$$

is nonnegative definite.

Proof: Suffices to show

$$B'\Sigma B - B'D(D'\Sigma^{-1}D)^{-1}D'B$$

is nonnegative definite, which follows from Lemma 1. $\square$

LEMMA **3.** Let B_i, D_i, $1 \le i \le m$, be $t \times p$ matrices. Let $\Sigma_1, \ldots, \Sigma_m$ be $t \times t$ positive definite matrices. Then

$$\left(\sum_{i=1}^{m} B_i'D_i\right)^{-1} \left(\sum_{i=1}^{m} B_i'\Sigma_i B_i\right) \left(\sum_{i=1}^{m} D_i'B_i\right)^{-1} - \left(\sum_{i=1}^{m} D_i'\Sigma_i^{-1}D_i\right)^{-1}$$

is nonnegative definite, assuming that the inverses exist.

Proof: Take $B' = (B_1'\ B_2'\ \ldots B_m')$; $D' = (D_1'\ D_2'\ \ldots D_m')$ and $\Sigma = \mathrm{diag}(\Sigma_1, \ldots, \Sigma_m)$. Note that

$$\begin{aligned} B'D &= \sum_{i=1}^{m} B_i'D_i \\ B'\Sigma B &= \sum_{i=1}^{m} B_i'\Sigma_i B_i \\ D'\Sigma^{-1}D &= \sum_{i=1}^{m} D_i'\Sigma_i^{-1}D_i \end{aligned}$$

and thus Lemma 3 follows from Lemma 2. $\square$